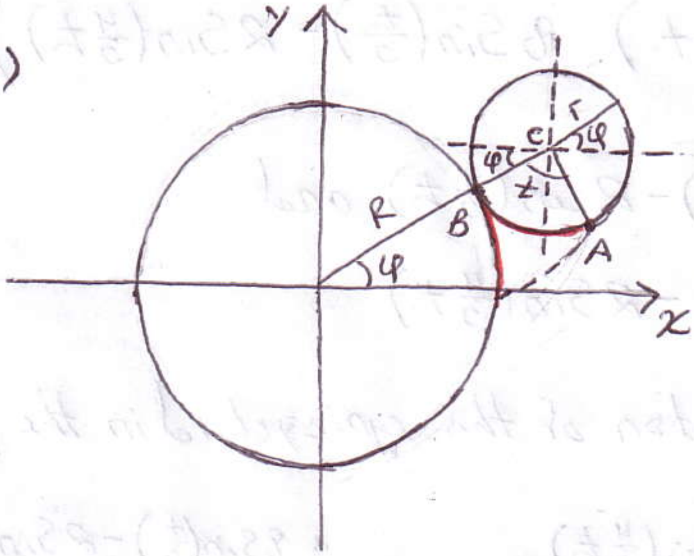
Solutions to Exam 2

1. a)



Let A be a fixed point on the small circle and B be a point that will touch the bigger circle after the small circle rolled by an angle t. We need to determine $A(t)$.

Notice that $A(t) = \vec{C} + \vec{AC}$ where

$\vec{C} = (R+r)(\cos\varphi, \sin\varphi)$ and

$\vec{AC} = r(\cos(\pi+\varphi+t), \sin(\pi+\varphi+t))$ (why?)

Observe that $\varphi R = tr$ (why?) so $\varphi = \frac{r}{R}t$.

Therefore,

$$\vec{AC} = r\left(\cos\left(\pi + \tfrac{R+r}{R}t\right), \sin\left(\pi+\tfrac{R+r}{R}t\right)\right) = r\left(-\cos\left(\tfrac{R+r}{R}t\right), -\sin\left(\tfrac{R+r}{R}t\right)\right)$$

$$= \left(-r\cos\left(\tfrac{R+r}{R}t\right), -r\sin\left(\tfrac{R+r}{R}t\right)\right) \text{ and}$$

$$\vec{C} = \left([R+r]\cos\left(\tfrac{r}{R}t\right), [R+r]\sin\left(\tfrac{r}{R}t\right)\right)$$

Thus,

$$A(t) = \left([R+r]\cos\left(\tfrac{r}{R}t\right) - r\cos\left(\tfrac{R+r}{R}t\right), [R+r]\sin\left(\tfrac{r}{R}t\right) - r\sin\left(\tfrac{R+r}{R}t\right)\right)$$

b) When $R=6$ and $r=2$, $A(t)$ becomes

$$\left(8\cos\left(\tfrac{t}{3}\right)-2\cos\left(\tfrac{4}{3}t\right),\ 8\sin\left(\tfrac{t}{3}\right)-2\sin\left(\tfrac{4}{3}t\right)\right).$$

c) let $s = 8\cos\left(\frac{t}{3}\right)-2\cos\left(\frac{4}{3}t\right)$ and

$t = 8\sin\left(\frac{t}{3}\right)-2\sin\left(\frac{4}{3}t\right)$

Then, the parameterization of this epicycloid in the plane is

$$(1,2,3)+\frac{8\cos\left(\frac{t}{3}\right)-2\cos\left(\frac{4}{3}t\right)}{\sqrt{18}}(-1,1,4)+\frac{8\sin\left(\frac{t}{3}\right)-2\sin\left(\frac{4}{3}t\right)}{\sqrt{2}}\cdot(1,1,0).$$

2. a) $f'(t) = (1, -4t, 3t^2)$

b) $Df(t)$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$. For fixed t, $Df(t)$ is a linear map.

$Df(t)(s) = (s, -4ts, 3t^2 s)$, where $s \in \mathbb{R}$.

In other words, $f'(t)$ is the velocity vector, while $Df(t)$ is the tangent line.

3. a) She should move in the direction of the gradient at $(0,1)$, $\nabla f(0,1)$.

Now $\nabla f(x,y) = (-6x, -2y)$. Therefore, $\nabla f(0,1) = (0,-2)$. Hence, the direction in which the woman should move is given by $u = \frac{(0,-2)}{\|(0,-2)\|} = \frac{(0,-2)}{2} = (0,-1)$.

b) The woman cannot move in a direction steeper than 1. Suppose that she moves at an angle θ to the gradient. Then $D_u f(0,1) = \|\nabla f(0,1)\| \cos\theta \le 1$ Since $\|\nabla f(0,1)\| = 2$, the maximal allowed steepness is achieved when $2\cos\theta = 1$, which happens when $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. Thus, the two possible solutions are

(1) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

(2) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

4. $J(f\circ g\circ h)(1,0) = Jf(g\circ h(1,0))\, Jg(h(1,0))\, Jh(1,0)$

a) Since $h(1,0) = (1,1)$, $g(h(1,0)) = g(1,1) = (0,2,1)$, we see that $J(f\circ g\circ h)(1,0) = Jf(0,2,1)\, Jg(1,1)\, Jh(1,0) =$

$$= \begin{pmatrix} 0 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 4 \\ 3 & 3 \end{pmatrix}$$

b) $D(f\circ g\circ h)(1,0)(u,v) = (12u + 4v,\ 3u + 3v)$

5. Let $f(x,y,z) = xyz$ be subject to the constraint S. then z is not independent of x and y. That is $z = z(x,y)$. Therefore $\left.\frac{\partial f}{\partial x}\right|_S = \frac{\partial}{\partial x}(xyz(x,y))$

Now $\frac{\partial}{\partial x}(xyz(x,y)) = yz + xy\frac{\partial z}{\partial x}$, where $\frac{\partial z}{\partial x}$ can be computed by taking the partial derivative of $x^4+y^4+z^4=1$ with respect to x.

$\frac{\partial}{\partial x}(x^4+y^4+z^4) = \frac{\partial}{\partial x}(1)$ or

$4x^3 + 4z^3\frac{\partial z}{\partial x} = 0$ Hence $\frac{\partial z}{\partial x} = \frac{-4x^3}{4z^3} = -\frac{x^3}{z^3}$.

With this result, we see that

$$\frac{\partial}{\partial x}(xyz) = yz + xy\frac{\partial z}{\partial x} = yz - \frac{x^4 y}{z^3}$$

To compute $\frac{\partial f}{\partial x}\Big|_S (0,0,1)$, just replace these values in place of x, y, and z: $0 \cdot 1 - \frac{0^4 \cdot 0}{1^3} = 0$.

6. We wish to minimize $x^2+y^2+z^2$ under the constraint $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ or $x^2 = a^2\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)$.

Let $f(y,z) = y^2+z^2+a^2\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)$

Now $\nabla f(y,z) = \left(2y - 2\frac{a^2}{b^2}y,\ 2z-2\frac{a^2}{c^2}z\right) = (0,0)$.

The only solution is $(y,z)=(0,0)$ (because $\frac{a^2}{b^2}\neq 1$).

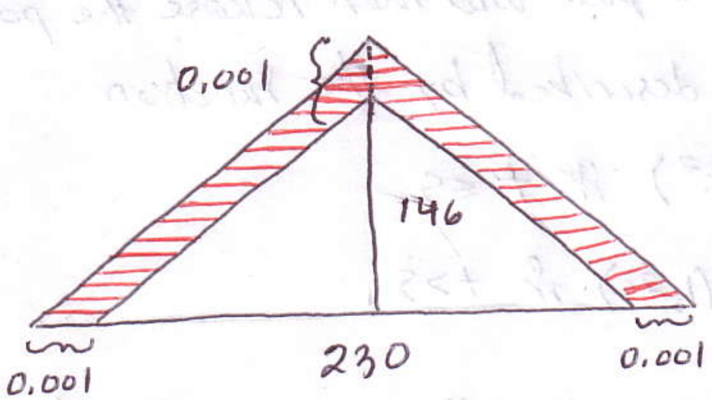
Now $H_f(y,z) = \begin{pmatrix} 2\,\frac{b^2-a^2}{b^2} & 0 \\ 0 & 2\,\frac{c^2-a^2}{c^2} \end{pmatrix}$ is positive-definite. Therefore f attains its minimum when $x^2 = f(0,0) = a^2$. Hence the closest points on the ellipsoid are $(-a,0,0)$ and $(a,0,0)$.

7. Let $g(x) = \sin(x)$ and $T(x,y) = 3x+2y$

Then $f = g \circ T$ where T is linear. Therefore, if $p(x)$ is the third-degree Taylor polynomial of g, $p \circ T$ is the third-order Taylor polynomial of f.

Now since the third-degree Taylor polynomial of $g(x) = \sin(x)$ is $p(x) = \sum_{n=0}^{3} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, the third-order Taylor polynomial of f, $P(x,y) = \sum_{n=0}^{3} \frac{(-1)^n (3x+2y)^{2n+1}}{(2n+1)!}$

8.



$$V = \frac{1}{3} h b^2$$

$$dV = \frac{\partial V}{\partial h}(146, 230) \cdot 0.001 + 2 \frac{\partial V}{\partial b}(146, 230) \cdot 0.001 =$$

$$= \left(\frac{\partial V}{\partial h} + 2 \frac{\partial V}{\partial b} \right) 0.001 = \left(\frac{1}{3} b^2 + \frac{4}{3} h b \right) \cdot 0.001 =$$

$$= \frac{0.001}{3} \left(b^2 + 4hb \right) = \frac{0.001}{3} 230 \left(230 + 4 \cdot 146 \right) \approx 62.4$$

Thus, 62.4 m^3 of gold was used.

9. $$\frac{\partial u}{\partial t} = sf(s+e^t, s-e^{-2t}) + ts\left(\frac{\partial f}{\partial x}(s+e^t, s-e^{-2t})e^t + \right.$$
$$\left. + \frac{\partial f}{\partial y}(s+e^t, s-e^{-2t}) \cdot 2e^{-2t}\right)$$

10. Let $u = (x^2+2y)$. Then $\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!} =$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2+2y)^{2n+1}}{(2n+1)!}$$

11. To arrive at the space station, you must first travel along the parabolic path and then release the pod for docking. Thus, your path is described by the function

$$p(t) = \begin{cases} (t, t^2) & \text{if } t \leq s \\ (s, s^2) + (t-s)(1, 2s) & \text{if } t > s \end{cases}$$

where s is the time when the pod is released.
Notice that docking takes place when $p(t) = g(t)$ or
$(s, s^2) + (t-s)(1, 2s) = (4 + \cos(\frac{\pi}{8}t), 8 - \sin(\frac{\pi}{8}t))$
This equation reduces to
$$t = 4 + \cos(\tfrac{\pi}{8}t)$$
$$2st - s^2 = 8 - \sin(\tfrac{\pi}{8}t)$$

Although the equation $t = 4 + \cos(\frac{\pi}{8}t)$ is difficult to solve

systematically, notice that when $t=4$, $4+\cos(\frac{\pi}{8}\cdot 4) =$
$= 4+\cos(\frac{\pi}{2}) = 4$. Therefore $t=4$ is a solution.
When $t=4$, the equation

$$2st - s^2 = 8 - \sin(\tfrac{\pi}{8}t)$$

becomes

$$8s - s^2 = 7$$

which reduces to

$$(s-1)(s-7) = 0$$

Since $s<t=4$, $s=1$ is the only solution.
Hence the pod should be released 1 year into the journey.
The pod will then float through space for additional 3 years before docking with the space station

12. Let $u(x) = \int_0^x g(s)\,ds$ and $v(x) = x$. Then

$F(x) = \int_0^{u(x)} f(v(x), t)\,dt$.

If we define $T(u,v) = \int_0^u f(v,t)\,dt$ and $S(x) = (u(x), v(x))$.
then $F(x) = T(S(x))$. Hence,

$$\frac{dF}{dx} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial x} = f(v(x), u(x))\cdot u'(x) + \int_0^{u(x)} \frac{\partial f}{\partial x}(v(x), t)\,dt \cdot$$

$$\cdot\, v'(x) = f\left(x, \int_0^x g(s)\,ds\right) g(x) + \int_0^{\int_0^x g(s)\,ds} \frac{\partial f}{\partial x}(x, t)\,dt.$$